**Weak Field Limit Dynamics**

Let’s examine our solution a little bit. So recall we find for the weak gravitational field, the following metric:



**Curvature**

Let’s do a little exploration of the metric. What is the curvature? Well, we have:



where,



(the bold letters stand for the matrices, ∙ stands for matrix multiplication, and the left arrow means those operators are acting on ) Let’s work these out using,



So,



So then our curvature is:



Well φ doesn’t depend on time, so:



Then, we can fill in the fact that:



So,



Note this agrees with what we found back in the Geometry file, namely that:



where in our case Λ = 0, and T is given by:



We can from this get the Ricci tensor, via Einstein’s equations:



We see that with Λ = 0:



And consistent with our assumption of slow moving/static matter Tαβ = ρ0c2δα0δβ0, we get:



well, for Rαβ, since I have gαβ,



which is:



Interesting. And the trace of this, Rαα = Tr(∙) does indeed reduce back to R.

**Physical Coordinates**

Now let’s look at physical time and physical space. Going back to our metric,



We said in the GR Geometry file that we could get physical coordinate increments from this:



Applied to our metric this means, (just locally, not globally, so that φ is roughly constant):



which we can express as:



Let’s say we switch to these coordinates. Then what does our metric look like? Should be Minkowskian.



where we note Xaa´ = ∂xa/∂xa´ = (Xa´a)-1. This will clearly give us:



So this is one example of how we can construct a local Minkowski metric. If φ were constant everywhere, then this transformation would be *exact*, and this would be the metric everywhere in terms of our physical coordinates. And so we’d actually have a completely flat space. This makes sense I guess because a constant potential means no force. Say we have a particle making some displacement dxα = (cdt, d**x**). What is physical velocity?



Physical energy-momentum is given by projecting coordinate momentum onto the physical axes/Minkowski space,



We can make some meandering and instructive manipulations here to write this in terms of physical quantities. First we’ll write coordinate momentum in terms of the coordinate and the particles own proper time τ (not to be confused with the rest frame’s proper time – see Geometry file). And then we’ll use the chain rule, introducing the physical time. Next we’ll observe that physical time is the time running in the instantaneous Minkowski frame. The particle’s proper time is 1/γ times the physical time, as in SR. Then we absorb the potential prefactors into the coordinate displacements to get the physical displacements. And in the last step we recognize these physical derivatives as the physical velocities.



which is what we ultimately expect. Note in particular that the physical energy here is the rest energy + kinetic energy. There is no ‘gravitational’ potential energy here. And that is because this instantaneous Minkowski frame is an inertial frame (instantaneously at rest, but freely falling) and so it doesn’t ‘see’ gravity.

**Equation of motion for particle in weak field**

Now let’s look at the equation of motion of a particle in such a weak field. As you’ll recall, the equation of motion is:



and there is no force, so the LHS is zero. Before we evaluate the RHS, let’s look at this quantity . Well, specifically let’s look at its magnitude. We know from SR that the magnitude (squared) of is –(mc)2. Let’s examine the consequences of this equality in the present context.



where **p** is the spatial part of the coordinate momentum vector. And so we get,



This just tells us how coordinate kinetic energy (+ rest energy) is related to coordinate momentum. With this in mind, let’s look at the equation of motion:



Let’s separate this equation into two parts – the energy and momentum part. So the zeroth component reads,



Now we’ll assume a classical sort of situation for simplicity so that p0 >> pi. In that case, just keeping the dominant terms in the equation,



And now use:



and since the metric is diagonal, the only non-zero terms are when m = 0 as well. So we have:



assuming the potential is time-independent. So we get an energy conservation equation of sorts:



I’m calling p0 the energy, but it’s not really the energy in any particular sense. Probably better just to call it the coordinate enegy. Now let’s look at the ith component of the coordinate momentum.



We can still assume that the α = β = 0 terms are the largest, in the classical limit. Then we have:



where



Again, only diagonal g elements are non-zero. So this comes to:



to first order in φ. And so then plugging this in we have:



which is of course N2L, in the context of a gravitational field φ.



In our case, the metric didn’t depend on time, and so it should be the case that:



(see Dynamics file) Now let’s look at p0, which is usually denoted as ‘the energy’, -E/c. So,



And so the energy E = -p0c is:



and this is of course the familiar classical expression for the energy of the particle (rest + kinetic + potential). So we just find that E is a constant of motion. We also found that p0 was approximately constant in time, in a previously boxed equation. But that was an approximation, while *this* is exact. And we can see the constancy of p0 follows from the constancy of p0 to first order, since we have:



Constancy of p0 and of p0 are equivalent as long as mc2 >> 2/2m >> mφ. In the Kinematics file, we said that ´ = -∙. What is this? Let’s say the local observer is at rest so that uα = (u0, 0). What is u0? Well,



So then the energy is given by:



to first order in φ. So we see that ´is just the rest + kinetic energy. This makes sense, as argued in the Kinematics file, because this energy is the energy w/r to a local inertial rest frame, and as such, does not ‘see’ gravity. Just for fun, let’s work out the energy if the observer is in this local inertial frame, but moving at the same speed as the observer. Then uα = pα/m, and we have:



and this makes sense because the observer is in the rest frame of the particle.

**Photon Trajectories**

Photon dynamics work out similarly. So start with,



We’ll look at the ‘energy’ conservation law. Observing once again that the metric doesn’t depend on time, we have υ0 will be conserved (see GR dynamics file). So,



As noted in the GR dynamics file, we can replace the υ in the the photon dynamics equations with p. So have,



This is a statement of conservation of energy for the photon. But it too, like the particle, gains and loses potential energy in a sense, as it climbs out of and falls into potential wells. Let’s look at photon red shift. One way to do this is to compare photon energies at different locations. Consider two points 1 and 2. We have that the energy of an observer in an at-rest inertial frame (freely falling) at these two points respectively is:



If the observer is at rest, then their four-velocity is given by:



as worked out above. So we have:



Now our conservation law says that the p1,0 = p2,0 = p0. So,



Equating p0 we have:



Now the energy measured in the rest frame is ´ = hν. Filling in our result for u0, we can see that:



Let the first point be at ∞, where φ1 = 0. Then we can conclude,



Another route to this formula is to simply say look at the time-dilation effects at different radii. Consider a wave at ∞ with period T*ν*. And consider what that same interval of time, Tr, will appear to be to an observer at radius r from the star. From the physical time formula thing, we have:



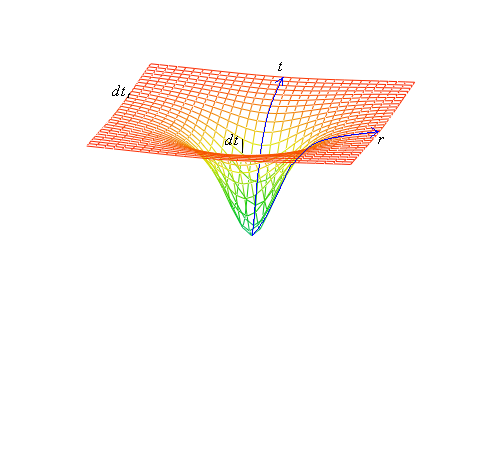
This formula is often written in terms of the wavelength:



And then the standard way to write it is:



As you can see, the further away from the potential the particle gets, the lower the frequency (longer the wavelength) becomes. Do note φ(r) is negative. This is also kind of consistent with the picture below:



As we climb the well, the radial coordinates get ‘smaller’, and so the same lengthed ‘thing’ would be measured longer the further from the well we go. Of course radial coordinate and physical radial distance are different things, but whatever.